UNIQ Problem Sheet 2014

This problem sheet is based on the material covered in lectures. You should attempt to do the problems during the periods of independent study and/or in your free time. On Thursday, you will have an hour-long tutorial, during which your solutions or your difficulties with finding them can be discussed. So, if you find some of the problems hard, don't despair! — they are meant to be and that's why we have tutorials. It is important, however, that you think about the problems and try to identify the key issues/concepts/calculations that you find difficult, so you can ask about them in the tutorial. Remember, it is largely up to you what happens in the tutorial — so you should come with an agenda.

You are welcome to discuss the problems with each other — it may be an especially good idea to discuss them with your tutorial partner, so the two of you can agree what you would like covered (there is never enough time, so you should decide what you want to discuss most).

Dimensional Analysis (Prof. Alex Schekochihin)

1. G. I. Taylor and the Bomb

Attempt this question after the first lecture on Dimensional Analysis

In the early autumn of 1940, during some of the most desperate days of the Battle of Britain, a Cambridge Professor of fluid dynamics G. I. Taylor was invited to lunch by an Imperial College Professor and Nobel-prize winner George Thomson, who was then chairman of the MAUD committee (MAUD = "Military Application of Uranium Detonation"). G. I. Taylor was told that it might be possible to produce a bomb in which a very large amount of energy would be released by nuclear fission (this was to become the atomic bomb). The crucial question was what would be the mechanical effect of such an explosion? G. I. Taylor's subsequent solution of this problem may be the most famous example of the application of dimensional analysis of all time. In this problem, you will work through some of his calculation.

Let us simplify the problem by assuming that
— a finite amount of energy E is released instantaneously at a point (i.e., we will ignore the radius r_0 of the volume where the initial energy release occurs at time $t = 0$, it will not be a relevant parameter);
— there results a *spherically symmetric* shock wave, with its front propagating according to some law $r_f(t)$, where r_f is the radius of the front.

Find $r_f(t)$ as a function of time t. Find also the velocity of the front $u_f(t)$ and the pressure $p_f(t)$ in the surrounding air just outside the front. The density ρ_0 of air before the explosion is given. *If you identify correctly what the governing parameters are (all of them are mentioned above), you should be able to use dimensional analysis to work out r_f, u_f and p_f with only constant dimensionless prefactors left undetermined.*

Based on the result you have obtained, will, in your opinion, making the bomb bigger (say doubling its size) makes much of a difference?

If you did not know the energy of the explosion E (classified!), but had a movie of the fireball, how would you estimate E? (When the Americans tested the bomb and released a series of high-speed photographs, G. I. Taylor estimated E and published the result, which caused much embarassment in the American government circles.)

2. Poiseuille Flow

Attempt this question after the second lecture on Dimensional Analysis

This example is also famous, and much more peaceful than the previous one. It was first worked out experimentally by H. Hagen (1839) and J. L. M. Poiseuille (1840) (working independently of each other) and later theoretically explained by G. G. Stokes (1845).

Consider a pipe of length l and diameter d. A pressure drop between the ends of the pipe, $p_1 - p_2$, is maintained to pump an incompressible fluid of viscosity μ through the pipe. Find the volumetric flow rate Q, i.e., the volume of the fluid that passes through any cross-section of the pipe per unit time.

If I double the diameter of the pipe, by what factor will Q change? What if I double the pressure contrast? And what if I double viscosity? Does the answer make sense? (Why does viscosity matter?) What if I double viscosity *and* cut the pipe length by half?

Hint. A judicious choice of governing parameters in this problem is d, μ and $(p_1 - p_2)/l$ — the pressure drop per unit length (think about why that is).

Now find the velocity U at which the fluid flows through the pipe.

Why do you think the density of the fluid does not matter here? Under what conditions would you expect it to start being an important parameter? (Think about the discussion in the lectures — what is the dimensionless number that controls this?)

Superconductivity (Prof. Andrew Boothroyd)

Attempt these questions after the first lecture on superconductivity

- **London penetration depth**. In 1935, F. and H. London, who were working in Oxford at the time, showed that a magnetic field applied parallel to the surface of a superconductor would penetrate a short distance into the superconductor. They derived the following equation for the variation of field $B(z)$ with distance z into the superconductor:

  $$\frac{\mathrm{d}^2 B(z)}{\mathrm{d}z^2} = \frac{B(z)}{\lambda^2}, \qquad (1)$$

  where

  $$\lambda^2 = \frac{m_\mathrm{e}}{\mu_0 n e^2}.$$

  Here, $m_\mathrm{e} = 9.11 \times 10^{-31}\,\mathrm{kg}$ is the mass of the electron, $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{Hm}^{-1}$ is the permeability of free space, $e = 1.60 \times 10^{-19}\,\mathrm{C}$ is the electronic charge, and n is the number of electrons per unit volume.

  If the field at the surface ($z = 0$) is B_0, show that the field decays exponentially into the superconductor. Make an estimate of λ and calculate the depth at which the field has decayed to $B_0/10$.

  [Hint: If you don't know how to solve eqn (1) directly, try substituting

  $$B(z) = C\exp(-z/\lambda) + D\exp(z/\lambda),$$

  and use the value of $B(z)$ at $z = 0$ and $z = \infty$ to determine the constants C and D.]

- **Superconducting wire**. A wire is to be made from the superconductor Nb_3Sn, which has a critical magnetic field $B_\mathrm{c} = 20\,\mathrm{Tesla}$. What is the minimum radius of the wire if it is to carry a current of $10^4\,\mathrm{A}$?

  What would happen if the current exceeded this value? What could be done to prevent damage to the wire?

  [Hint for the first part: The magnetic field outside a wire carrying a current I is given by $B(r) = \mu_0 I/(2\pi r)$, where r is the distance from the centre of the wire.]

Relativity (Prof. Steve Biller)

The nearest potentially inhabitable planet observed so far orbits the star Gliese 581, at a distance of 20 lightyears (*i.e.* 20 times the distance light travels in one year) from earth. The fastest spaceship ever launched is the New Horizons probe, currently *en route* to Pluto, which has achieved a speed in excess of 50000 km/hr using a boost from Jupiter's gravity. How long would it take such a rocket to reach this star?

As part of their summer programme, one team of UNIQ physics students invents a new propulsion system capable of travelling at 90% the speed of light. Another team of students builds the rocket, which is 100m in length, while a third team offers to fly it to Gliese 581 in the hope of finding better financial support for students elsewhere in the Universe.

According to observers on Earth:
How long will it take the spacecraft to reach the star?
What is the length of the rocket while it is travelling there?

According to the students in the rocket:
How long will it take them to reach the star?
How do they explain this given their speed?

A future group of concerned students back on earth (concerned because they are still paying higher tuition fees!) build a huge telescope to point towards the star. When should they look through it to see the spaceship arrive at its destination?

In the Earth's frame, depict the spacecraft journey and the light received by the telescope on earth using a space-time diagram.

BONUS QUESTION: On the way back, one student decides to tap into the ship's matter-antimatter power generator to keep their iPad charged. The energy estimated to do this corresponds to 12 kW-hr per year. Assuming the generator is 100% efficient, how much fuel is used up? How would the students on Earth calculate this?

Astrophysics **(Prof. Adrianne Slyz)**

How do you weigh a galaxy? Measuring the speed of rotation of stars in the outer regions of galaxies tells us about their mass. Let's investigate this:

(a) Derive an expression for the circular velocity, v, of a star at radius r, orbiting around a mass M for the case where its acceleration is caused by the gravitational pull of M. (Recall Newton's law for gravitation: $F = GMm/r^2$.)

(b) Let's assume stars dominate the visible mass in a galaxy. A very approximate model for the stars in a galaxy disk is a sphere of uniform density, ρ_0, with radius R. (In fact, disk galaxies are clearly not spherical, but let's ignore that for now!). We can calculate the total mass of stars within a radius r as $M(r) = \rho_0 V(r)$, where $V(r)$ is the volume enclosed within the radius r.

Using the formula for the circular velocity you found in (a), derive an analytic expression for the velocity of a star in the galaxy disk as a function of r, in two regimes: (1) for $r < R$ inside the galaxy disk, and (2) for $r > R$ outside the disk. This relationship between the velocity and radius is called the rotation curve of a galaxy.

(c) Assume $M = 7.5 \times 10^9\ M_{\rm sun}$ and $R = 6$ kpc. Using your formula for v as a function of R that you derived in part (b), sketch the galaxy's rotation curve from $r = 0$ to $r = 60$ kpc, working out the rotation speed v at radius $R = 6$ kpc.

(d) It turns out that the observed rotation curve is different from the one you sketched in (c). At $R = 6$ kpc it has a value of ~ 150 km/s. Approximately how much mass needs to be enclosed within R to give this circular velocity? So, given how much stellar mass there is, how much extra 'invisible matter' is required? This extra invisible matter is what we call dark matter.

(e) It also turns out that the rotation curve of the galaxy is flat out to at least 60 kpc. To get a flat rotation curve, how must the density of dark matter scale with radius? Roughly what fraction of the galaxy's mass within 60 kpc would be in the form of dark matter?

Useful constants:
$G = 6.67 \times 10^{-11}\ {\rm m^3\ kg^{-1}\ s^{-2}}$ $\quad$ $1\ {\rm pc} = 3.086 \times 10^{16}$ m $\quad$ $M_{\rm sun} = 1.9889 \times 10^{30}$ kg